Singularity Theorems in Cosmology: Searching for a more accessible approach

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We explore the mechanics of the singularity theorems first formulated by Hawking and Penrose. We focus on the role of the expansion of Null Geodesic Congruences, and how this can be used to illustrate the conditions in which cosmologies fail the test of those theorems. We also discus how this pertains to the generation of an Inflationary region in de Sitter space consistent with current observations of a cosmological constant.

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I. INTRODUCTION

Cosmic Inflation, as a model of our universe, has been very useful in explaining the events that followed it, but it turns out to be a difficult problem to determine how to get a region of space to start cosmic inflation. In fact, singularity theorems formulated in General Relativity have been used to rule out a classical beginning to Cosmic Inflation all together [1] [2]. This presents a difficulty, because observations so far are consistent with an inflationary origin. Of particular interest are recent observations indicating the presence of dark energy, an energy state very similar to the theoretical energy that would drive Cosmic Inflation. If this dark energy represents a fundamental cosmological constant, it could imply some interesting constraints on our universe. One would be that there is a finite extent to what we can observe of the universe, and with that finite extent could well come a finite number of possible states. This is a result of the accelerated expansion a cosmological constant would impose on the universe, pushing all objects beyond the horizon of what we can see. The cosmological constant can be seen as creating space, and the horizon is where the aggregate creation of space causes the distance from the observer to increase at the speed of light, effectively cutting off anything outside of that distance.. If the universe is finite with a finite number of states, then every state will recur, with a particular average rate, one that should be given by the energy of the state and the temperature of the Hawking radiation emitted by the horizon.

Hawking and Penrose originally put forth their singularity theorems to show that classical General Relativity predicts its own demise under very general circumstances. The theorems point out contradictions in proposed space-times, and are very dependent on energy conditions, which can be seen as constraints on the energy density and how it is related to pressure. As such, singularity theorems are a useful tool to check if a loosely constrained idea is consistent with General Relativity. But, when we use singularity theorems to reject proposed space-times, it takes considerably more effort to find the critical constraint that enforces the failure or could be relaxed to allow the solution. The purpose of this paper is to explore the constraints imposed by singularity theorems, and to suggest avenues by which they can be circumvented.

II. SINGULARITY THEOREM BASICS:

For the purposes of singularity theorems in Cosmic Inflation, the key concept is the rate of change of the expansion of Null Geodesic Congruences (NGC). An NGC is a family of curves in space time that behave like light. In particular, we only need to consider NGC's that can be represented as spherical shells of light directed inward or outward radially from their center. We often do not want to imagine actual photons traveling on those trajectories, because real photons imply an energy density that would itself affect the curvature of the space-time.

The expansion θ is a measure of how quickly the distances are increasing between the various null rays that make up the NGC, [3] (derived from the Raychaudhuri's equation). In spherically symmetric cases without shear and twist, the equation governing θ, as derived from the Raychaudhuri's equation [3], can be written as

$$\frac{\partial\theta}{\partial\lambda} = -\frac{1}{2}\theta^2 - R_{\mu\nu}v^\mu v^\nu \tag{1}$$

where λ is the affine parameter, $R_{\mu\nu}$ is the Ricci tensor, and v^μ is the null vector for the NGC. In space times with shear or twist, there are additional terms that always add negatively to the right side of the equation. The affine parameter is a general way of parameterizing geodesics; for non-null geodesics, the affine parameter is the proper time.

A plane wave in Minkowski space would have a θ of 0, while the θ for spherical wave is inversely proportional to radius, positive for outward going NGC's, negative for inward. For an ingoing sphere that reaches the center

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and becomes and outgoing sphere, θ runs to negative infinity at the moment of crossing, and down from positive infinity after.

In the absence of space-time curvature, a positive θ will tend toward zero as the NGC's radius tends toward infinity. A Negative θ runs off to negative infinity as radius goes to zero, where the NGC crosses itself, and then run down from positive infinity as the NGC re-expands. Starting with an initial negative θ_0, the crossing happens in an affine length of $\frac{2}{\theta_0}$. As this equation is time-symmetric, this also means that positive θ will have a point of crossing, with infinite positive expansion, a similar finite affine length in the past. With curvature, things can become more complicated, but the core of the various singularity theorems is that, for normal matter, $-R_{\mu\nu}v^{\mu}v^{\nu}$ is always non-positive. Normal matter is defined here as mater obeying the weak energy condition; for energy density this is $\rho \geq 0$, and for pressure, $p \geq -\rho$. De Sitter space, a highly symmetric space that Cosmic Inflation approximates, is the extreme case where $-R_{\mu\nu}v^{\mu}v^{\nu}$ is zero for spherical NGC's. The important result of this is that, while expanding NGC's can easily turn into contracting NGC's, the only way to go from contracting to expanding without crossing is to violate the weak energy condition.

III. APPLICATIONS

With recent evidence for cosmic acceleration, and taking Hawking radiation in such a universe seriously, there is a very attractive possibility of viewing inflation as a rare quantum fluctuation of an equilibrium state. In particular, there could be a low but non-zero probability for microscopic levels of Hawking radiation to "boil" off the horizon and eventually concentrate in a patch large enough for Cosmic Inflation to begin in classically. However, the singularity theorems tell us that if we treat the waves classically after their initial appearance, there is no way to form an inflating region. As such, we want to find a way around the singularity theorems in a believable way.

A. Pure de Sitter Space

De Sitter Space is a highly symmetric solution to Einstein's equations. It is isotropic and homogeneous, and reflects an empty universe with a positive cosmological constant (a negative one defines Anti de Sitter Space). Closed (k=1) de Sitter and open (k=-1) de Sitter spaces have, respectively, an element of positive and negative curvature that changes over time, while flat (k=0) de Sitter space is unchanging for all time. We consider de Sitter space important largely because we seem to see our universe tending toward it now, and the stage of Cosmic Inflation theorized to have happened at the beginning of our universe is almost de Sitter. These solutions to Einstein's equations are typically expressed by the line element

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{2}$$

where $a(t)$ is the scale factor, k is the curvature, and Ω denotes solid angle.

The scale the dominates de Sitter space is set by the value of $H = \frac{\dot{a}}{a}$, known as the Hubble value, and its inverse, H^{-1}, the Hubble length. The dependence of the Hubble value is most clearly seen from the Freedman equations

$$H^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} \tag{3}$$

$$\dot{H} = -4\pi(\rho + p) + \frac{k}{a^2} \tag{4}$$

where ρ and p are the energy density and pressure, respectively. True de Sitter space is filled uniformly with an energy density for which pressure is the opposite of density. In a flat de Sitter space, inward directed spherical NGC's that have radii larger than the Hubble length are expanding. This is a simple and central feature to de Sitter space, that light beyond the horizon never travels inward. The expansion equation tells us that these NGC's would have to have had an infinite positive expansion a finite affine length in the past, as they have a finite positive expansion now. This could be viewed as requiring a white-hole singularity or a big bang singularity, but there are other ways out. We find that the affine parameter is dilated as we trace the NGC's back to the Hubble radius, such that they do not reach that affine length at past infinity, and in fact originate at the Hubble radius. Actual particles following these trajectories are typically ruled out with initial conditions, as they require infinite energy in the infinite past [12].

The typical way of depicting de Sitter space is a hyperboloid diagram with a contracting region followed by and expanding region (this is a time slicing that gives a closed universe, see figure). In the contracting phase, there are outward directed NGC's that are converging, while the expanding phase has inward directed NGC's that are expanding. As it turns out, this is not a contradiction by the singularity theorems, because the NGC's wrap around the curvature of the space, and we can change our perspective so that the NGC's directed away from one pole can be seen as directed toward the other. The critical set that defines this property are the NGC's with zero expansion, which appear as parallel diagonal lines on opposite sides of the the hyperbolic de Sitter diagram. In a sense, the NGC's to one side of this dividing line are contracting, while the NGC's on the other side are expanding.

B. Inflation Basics

Cosmic Inflation refers to a short period of time near the beginning of the universe when the energy density was dominated by the potential energy of a scalar field. This brings the pressure very close to the opposite of density, such that the universe mimics de Sitter space for the duration. This smooths the universe out, pushing initial inhomogeneities and signs of curvature beyond the horizon. Inflation was originally conceived as emerging from a singular Big Bang, which would have been very inhomogeneous, as a way of explaining the observed large scale homogeneity of the universe. It was not until more recently that the idea of inflation itself as a starting point has been raised.

The mimicry of de Sitter space, however, is not perfect, and the (relatively) slow rate of decrease in the scalar potential gives rise to a nearly scale invariant spectrum of density fluctuations. The period of inflation ends with an event known as reheating, where the energy of the scalar field is converted thermally into other particles. Later, as the universe cools, the initial density fluctuations start the seeds of large scale structure. Though Cosmic Inflation invokes an as yet undiscovered particle field, astronomical tests indicate that, at the very least, something that looks like Inflation must have preceded the earliest times we can describe with confidence.

C. Cosmic Inflation from Late de Sitter Space

There is a definite problem, however, if you imagine an inflating cosmology springing from something with a more leisurely expansion rate (something with a smaller Hubble constant/larger Hubble radius), rather than existing for all past eternity. Inward directed NGC's that are outside the Hubble radius of Inflation but inside the Hubble radius of the state that preceded inflation would have gone from contracting to expanding without crossing. If we start with an empty, flat de Sitter space with a cosmological constant similar to the one theorize to account for the Cosmic Acceleration that has been observed in recent years, the Hubble radius would be larger than the observable universe today. The Hubble radius for Cosmic Inflation is microscopic, smaller than the length scales yet probed by particle accelerators. While we have evidence of going from Inflation eventually to 'acceleration,' we will consider what would happen if we wanted to go from 'acceleration' to Inflation. During the time period of Acceleration, NGC's that have a smaller radius than the Acceleration Hubble radius are contracting (negative θ). During inflation, however, all NGC's that are outside of Inflation's Hubble radius (but inside the inflating region) are expanding (positive θ). If we trace these inward directed but expanding NGC's back to the transition between the two periods, it seems that they must have been inward directed and contracting NGC's during Acceleration. This would be a violation of the expansion equation, because we have gone from negative θ to positive θ without crossing.

A closed (positive curvature) de Sitter space represents a version of inflation that seems to bypass the singularity theorems. It is then easy to cut off this de Sitter digram above and bellow the turnaround point, and match those to reheating like events and patches that look like our universe in the future, and some other closed universe in the past. This would give a cosmology that looks like our universe and sprang from something that looks like our future. The problem is that this history of our universe is unlikely [4], if even possible. If fact, it would be more likely for the universe to fluctuate into a reverse big bang, or big crunch, in a way that gives us inside a reversed arrow of time. The telling problem here is that the entropy of the closed de Sitter space at the turnaround will be microscopic, but still encompasses the entropy of the entire universe. As such, as unlikely as it is that we are living perfectly backward, with all of our observations conspiring to deceive us, it is still more likely than producing that turning point in this way.

D. The Budding Universe

If, on the other hand, one could start inflation in a patch of space just large enough to support inflation, leaving the rest untouched, the microscopic energy of that state would mean such a process would be greatly favored. This is because the probability of a fluctuation goes up as its total energy, or mass, goes down [5]. Can the contracting phase of de Sitter space then be matched onto a small piece of a larger flat space? The answer appears to be no, or at least not without violating the weak energy condition. The critical zero expansion NGC's could conceivably be matched on in a fashion that allows them to cross, avoiding contradiction, but doing so makes it necessary that near by NGC's must also cross at an earlier time, and so on, making it impossible to connect the spaces. There have been attempts at getting around this be invoking a quantum tunneling processes, but the theories put forth seem to require an initial singularity non the less [6].

Short term violations of the energy conditions, on the other hand, could lead to budding universes, but how they would connect to the original is a bit hazy. The minimal amount of energy condition violation necessary is actually a boundary region of sorts; if we start with some concentration of energy density, then just enough space with $w < -1$ to turn the in falling NGC's near the horizon from contracting to expanding is all that is required initially. Once the budding universe gets going, we just need a way of stopping in falling NGC's from the outside from leaking into the bud, which would make them expanding NGC's when the bud becomes bigger than the radius of the connecting space. This could be accomplished if the connection between the bud and the rest of the universe is a standard Einstein-Rosen bridge, also known as a non-traversable worm hole (see figure). It is clear what this would look like from the outside, but the view from the inside of the bud would probably have to be determined by a "Swiss cheese" model [9]. Various specific proposals for budding, or baby, universes, have appeared in the literature [10].

The clearest path to a viable version of this model seems to be through numerical efforts. Using a spherically symmetric version of the Einstein tensor [11], further research could be done by creating space-time metrics that result in budding, inflating universes, and examining the resulting Einstein tensors, which would yield the required stress-energy tensors. The space of possible metrics could then be explored to find how much violation of the energy conditions is necessary to make a budding, inflationary universe, so as to provide a framework for field theories that might allow such violations.

E. From Dark Energy to Big Rip, Big Bang, and back again

Another approach [7] argues for an energy condition violating quantum fluctuation that completely replaces Cosmic Inflation, but requires that fluctuation to be on the same scale as the observable universe, running into the same problem as that contracting and expanding closed inflation idea.

Long term violations of the energy conditions, in the form of a constant $w < -1$, result in more unusual predictions [8]. Such an equation of state is that of a dark energy that increases it's density as the universe expands, and in fact leads to a divergent energy in a finite time. Allow for quantum interactions, and you can imagine a universe of clockwork cycles, where the dark energy we may see today runs up in density in several billion years, scattering all matter in a Big Rip. If the dark energy then somehow decays down to nearly nothing by producing mater particles in a new Big Bang, we could then closed the loop, ending up again in a universe like ours. This would be significantly different than the other, much more carefully thought out cyclic models [13] in so much as it never even approaches a turn around point in terms of the scale factor.

Beyond just tampering with the energy conditions, there remains the possibility of looking to modifications of Einstein's Relativity. However, the present state of Inflationary theory is not likely to put many constraints on such modifications, so intervention from that quarter will probably be serendipitous, if at all.

Acknowledgments

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