Field equations for a general radially symmetric scalar field. The line element is

$$ds^2 = -f(r,t)dt^2 + h(r,t)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

The ϕ(scalar field) variation equation is

$$-\frac{1}{f}\frac{\partial^2\phi}{\partial t^2}+\frac{1}{h}\frac{\partial^2\phi}{\partial r^2}+\left(\frac{3}{2}\frac{\partial f}{\partial t}-\frac{1}{2}\frac{f}{h}\frac{\partial h}{\partial t}\right)\frac{\partial\phi}{\partial t}+\left(\frac{2}{r}h+\frac{3}{2}\frac{\partial h}{\partial r}+\frac{1}{2}\frac{h}{f}\frac{\partial f}{\partial r}\right)\frac{\partial\phi}{\partial r}-\frac{\partial U}{\partial\phi}=0 \tag{2}$$

Building up the field equations. From here on, $'$ denotes a partial derivative with respect to t, and $_r$ denotes a partial derivative with respect to r.

$$\Gamma^t_{\mu\nu} = \begin{pmatrix} \frac{1}{2}\frac{f'}{f} & \frac{1}{2}\frac{f_r}{f} & 0 & 0 \\ \frac{1}{2}\frac{f_r}{f} & \frac{1}{2}\frac{h'}{f} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

$$\Gamma^r_{\mu\nu} = \begin{pmatrix} \frac{1}{2}\frac{f_r}{h} & \frac{1}{2}\frac{h'}{h} & 0 & 0 \\ \frac{1}{2}\frac{h'}{h} & \frac{1}{2}\frac{h_r}{h} & 0 & 0 \\ 0 & 0 & -\frac{r}{h} & 0 \\ 0 & 0 & 0 & -\frac{r}{h}\sin^2\theta \end{pmatrix} \tag{4}$$

$$\Gamma^\theta_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{pmatrix} \tag{5}$$

$$\Gamma^\phi_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{pmatrix} \tag{6}$$

$$\partial_\lambda\Gamma^\lambda_{\mu\nu} = \begin{pmatrix} \frac{1}{2}\left(\frac{f'}{f}\right)' + \frac{1}{2}\left(\frac{f_r}{h}\right)_r & \frac{1}{2}\left(\frac{f_r}{f}\right)' + \frac{1}{2}\left(\frac{h'}{h}\right)_r & 0 & 0 \\ \frac{1}{2}\left(\frac{f_r}{f}\right)' + \frac{1}{2}\left(\frac{h_r}{h}\right)' & \frac{1}{2}\left(\frac{h'}{f}\right)' + \frac{1}{2}\left(\frac{h_r}{h}\right)_r & 0 & 0 \\ 0 & 0 & -\frac{1}{h} & 0 \\ 0 & 0 & 0 & -\frac{1}{h}\sin^2\theta - \cos^2\theta + \sin^2\theta \end{pmatrix} \tag{7}$$

$$\partial_\nu\Gamma^\lambda_{\mu\lambda} = \begin{pmatrix} \frac{1}{2}\left(\frac{f'}{f}\right)' + \frac{1}{2}\left(\frac{h'}{h}\right)' & \frac{1}{2}\left(\frac{f'}{f}\right)_r + \frac{1}{2}\left(\frac{h'}{h}\right)_r & 0 & 0 \\ \frac{1}{2}\left(\frac{f'}{f}\right)_r + \frac{1}{2}\left(\frac{h'}{h}\right)_r & \frac{1}{2}\left(\frac{f_r}{f}\right)_r + \frac{1}{2}\left(\frac{h_r}{h}\right)_r - \frac{2}{r^2} & 0 & 0 \\ 0 & 0 & -\csc^2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

$$\Gamma^\lambda_{\rho\lambda}\Gamma^\rho_{\mu\nu} = \begin{pmatrix} \frac{1}{4}\left(\frac{f'^2}{f^2}+\frac{f'h'}{fh}+\frac{f_r^2}{fh}+\frac{f_rh_r}{h^2}\right)+\frac{f_r}{rh} & \frac{1}{4}\frac{f'f_r}{f^2}+\frac{1}{2}\frac{f_rh'}{fh}+\frac{1}{4}\frac{h'h_r}{h^2}+\frac{h'}{rh} & 0 & 0 \\ \frac{1}{4}\frac{f'f_r}{f^2}+\frac{1}{2}\frac{f_rh'}{fh}+\frac{1}{4}\frac{h'h_r}{h^2}+\frac{h'}{rh} & \frac{1}{4}\left(\frac{f'h'}{f^2}+\frac{h'^2}{fh}+\frac{h_r^2}{h^2}+\frac{f_rh_r}{fh}\right)+\frac{h_r}{rh} & \frac{\cot\theta}{r} & 0 \\ 0 & \frac{\cot\theta}{r} & -\frac{1}{2}\frac{rf_r}{fh}-\frac{1}{2}\frac{rh_r}{h^2}-\frac{2}{h} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\sin^2\theta\left(\frac{rf_r}{fh}+\frac{rh_r}{h^2}+\frac{4}{h}\right)-\cos^2\theta \end{pmatrix} \tag{9}$$

$$\Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\lambda} = \begin{pmatrix} \frac{1}{4}\frac{f'^2}{f^2}+\frac{1}{2}\frac{f_r^2}{fh}+\frac{1}{4}\frac{h'^2}{h^2} & \frac{1}{4}\frac{f'f_r}{f^2}+\frac{1}{2}\frac{f_rh'}{fh}+\frac{1}{4}\frac{h'h_r}{h^2} & 0 & 0 \\ \frac{1}{4}\frac{f'f_r}{f^2}+\frac{1}{2}\frac{f_rh'}{fh}+\frac{1}{4}\frac{h'h_r}{h^2} & \frac{1}{4}\frac{f_r^2}{f^2}+\frac{1}{2}\frac{h'^2}{fh}+\frac{1}{4}\frac{h_r^2}{h^2}+\frac{2}{r^2} & \frac{\cot\theta}{r} & 0 \\ 0 & \frac{\cot\theta}{r} & -\frac{2}{h}+\cot^2\theta & 0 \\ 0 & 0 & 0 & -\frac{2}{h}\sin^2\theta-2\cos^2\theta \end{pmatrix} \tag{10}$$

$$\begin{aligned} R_{tt} &= \frac{1}{2}\left(\frac{f_r}{h}\right)_r - \frac{1}{2}\left(\frac{h'}{h}\right)' + \frac{1}{4}\left(\frac{f'h'}{fh}-\frac{f_r^2}{fh}+\frac{f_rh_r}{h^2}-\frac{h'^2}{h^2}\right)+\frac{f_r}{rh} \\ R_{rr} &= \frac{1}{2}\left(\frac{h'}{f}\right)' - \frac{1}{2}\left(\frac{f_r}{f}\right)_r + \frac{1}{4}\left(\frac{f'h'}{f^2}-\frac{h'^2}{fh}+\frac{f_rh_r}{fh}-\frac{f_r^2}{f^2}\right)+\frac{h_r}{rh} \\ R_{rt} &= \frac{h'}{rh} \\ R_{\theta\theta} &= 1-\frac{1}{h}-\frac{r}{2}\frac{f_r}{fh}-\frac{r}{2}\frac{h_r}{h^2} \\ R_{\varphi\varphi} &= \sin^2\theta\left(1-\frac{1}{h}-\frac{r}{2}\frac{f_r}{fh}-\frac{r}{2}\frac{h_r}{h^2}\right) = R_{\theta\theta}\sin^2\theta \end{aligned}$$

$$R = -f^{-1}R_{tt}+h^{-1}R_{rr}+2r^{-2}R_{\theta\theta} \tag{11}$$

$$\begin{aligned} R \quad = \quad & -\frac{1}{2f}\left(\frac{f_r}{h}\right)_r + \frac{1}{2f}\left(\frac{h'}{h}\right)' + \frac{1}{2h}\left(\frac{h'}{f}\right)' - \frac{1}{2h}\left(\frac{f_r}{f}\right)_r - \frac{f_r}{rfh}+\frac{h_r}{rh^2} \\ & +\frac{2}{r^2}\left(1-\frac{1}{h}-\frac{r}{2}\frac{f_r}{fh}-\frac{r}{2}\frac{h_r}{h^2}\right) \end{aligned}$$

$$R = \frac{h''}{fh}-\frac{f_{rr}}{fh}-\frac{1}{2}\frac{h'^2}{fh^2}-\frac{1}{2}\frac{h'f'}{hf^2}+\frac{1}{2}\frac{f_r^2}{hf^2}+\frac{1}{2}\frac{f_rh_r}{fh^2}-2\frac{f_r}{rfh}+\frac{2}{r^2}-\frac{2}{r^2h} \tag{12}$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right) \tag{13}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{M_p^2}T_{\mu\nu} \tag{14}$$

The easiest non-trivial equation is the rt equation, as $g_{\mu\nu}$ is diagonal.

$$\frac{h'}{rh} = \frac{8\pi}{M_p^2}\phi'\phi_r \tag{15}$$

All other off diagonal equations are identically zero.

$$\begin{aligned}
R_{tt} - \frac{1}{2}g_{tt}R \quad &= \frac{1}{2}\left(\frac{f_r}{h}\right)_r - \frac{1}{2}\left(\frac{h'}{h}\right)' + \frac{1}{4}\left(\frac{f'h'}{fh} - \frac{f_r^2}{fh} + \frac{f_rh_r}{h^2} - \frac{h'^2}{h^2}\right) + \frac{f_r}{rh} \\
&+\frac{f}{2}\left(\frac{h''}{fh} - \frac{f_{rr}}{fh} - \frac{1}{2}\frac{h'^2}{fh^2} - \frac{1}{2}\frac{h'f'}{hf^2} + \frac{1}{2}\frac{f_r^2}{hf^2} + \frac{1}{2}\frac{f_rh_r}{fh^2} - 2\frac{f_r}{rfh} + \frac{2}{r^2} - \frac{2}{r^2h}\right) \\
&= \frac{f}{r^2}\left(1 - \frac{1}{h}\right)
\end{aligned}$$

$$\begin{aligned}
R_{rr} - \frac{1}{2}g_{rr}R \quad &= \frac{1}{2}\left(\frac{h'}{f}\right)' - \frac{1}{2}\left(\frac{f_r}{f}\right)_r + \frac{1}{4}\left(\frac{f'h'}{f^2} - \frac{h'^2}{fh} + \frac{f_rh_r}{fh} - \frac{f_r^2}{f^2}\right) + \frac{h_r}{rh} \\
&-\frac{h}{2}\left(\frac{h''}{fh} - \frac{f_{rr}}{fh} - \frac{1}{2}\frac{h'^2}{fh^2} - \frac{1}{2}\frac{h'f'}{hf^2} + \frac{1}{2}\frac{f_r^2}{hf^2} + \frac{1}{2}\frac{f_rh_r}{fh^2} - 2\frac{f_r}{rfh} + \frac{2}{r^2} - \frac{2}{r^2h}\right) \\
&= \frac{h_r}{rh} + \frac{f_r}{rf} + \frac{1-h}{r^2}
\end{aligned}$$

$$\begin{aligned}
R_{\theta\theta} - \frac{1}{2}g_{\theta\theta}R \quad &= 1 - \frac{1}{h} - \frac{r}{2}\frac{f_r}{fh} - \frac{r}{2}\frac{h_r}{h^2} \\
&-\frac{1}{2}r^2\left(\frac{h''}{fh} - \frac{f_{rr}}{fh} - \frac{1}{2}\frac{h'^2}{fh^2} - \frac{1}{2}\frac{h'f'}{hf^2} + \frac{1}{2}\frac{f_r^2}{hf^2} + \frac{1}{2}\frac{f_rh_r}{fh^2} - 2\frac{f_r}{rfh} + \frac{2}{r^2} - \frac{2}{r^2h}\right) \\
&= r^2\left(-\frac{1}{2}\frac{h''}{fh} + \frac{1}{2}\frac{f_{rr}}{fh} + \frac{1}{4}\frac{h'^2}{fh^2} + \frac{1}{4}\frac{h'f'}{hf^2} - \frac{1}{4}\frac{f_r^2}{hf^2} - \frac{1}{4}\frac{f_rh_r}{fh^2} + \frac{1}{2}\frac{f_r}{rfh}\right)
\end{aligned}$$

The field equations for from D'Inverno pg 187 are

$$R_{tt} - \frac{1}{2}g_{tt}R = \frac{fh_r}{rh^2} + \frac{f}{r^2}\left(1 - \frac{1}{h}\right) \tag{16}$$

$$R_{rr} - \frac{1}{2}g_{rr}R = \frac{f_r}{rf} - \frac{h}{r^2}\left(1 - \frac{1}{h}\right) \tag{17}$$

$$R_{\theta\theta}-\frac{1}{2}g_{\theta\theta}R = r^2\left(-\frac{1}{2}\frac{h''}{fh}+\frac{1}{2}\frac{f_{rr}}{fh}+\frac{1}{4}\frac{h'^2}{fh^2}+\frac{1}{4}\frac{h'f'}{hf^2}-\frac{1}{4}\frac{f_r^2}{hf^2}-\frac{1}{4}\frac{f_rh_r}{fh^2}+\frac{1}{2}\frac{f_r}{rfh}-\frac{1}{2}\frac{h_r}{rh^2}\right) \tag{18}$$

The tr equation remains unchanged:

$$\frac{h'}{rh} = \frac{8\pi}{M_p^2}\phi'\phi_r \tag{19}$$

The tt equation is

$$\frac{fh_r}{rh^2}+\frac{f}{r^2}\left(1-\frac{1}{h}\right) = \frac{8\pi}{M_p^2}\left(\phi'^2-\frac{1}{2}f^2\phi'^2+\frac{1}{2}fh\phi_r^2+fV(\phi)\right) \tag{20}$$

and the rr equation is

$$\frac{f_r}{rf}-\frac{h}{r^2}\left(1-\frac{1}{h}\right) = \frac{8\pi}{M_p^2}\left(\phi_r^2+\frac{1}{2}fh\phi'^2-\frac{1}{2}h^2\phi_r^2-hV(\phi)\right) \tag{21}$$

The $\phi\phi$ and $\theta\theta$ equations are identical:

$$-\frac{h''}{fh}+\frac{f_{rr}}{fh}+\frac{1}{2}\frac{h'^2}{fh^2}+\frac{1}{2}\frac{h'f'}{hf^2}+\frac{1}{2}\frac{f_r^2}{hf^2}-\frac{1}{2}\frac{f_rh_r}{fh^2}+\frac{f_r}{rfh}-\frac{h_r}{rh^2} = \frac{8\pi}{M_p^2}\left(f\phi'^2-h\phi_r^2-2V(\phi)\right) \tag{22}$$

Thus, for purposes of calculating the "next step" we have:

$$\frac{\partial\phi}{\partial t} = \phi' \tag{23}$$

$$\frac{\partial h}{\partial t} = \frac{8\pi}{M_p^2}rh\phi'\phi_r = h' \tag{24}$$

$$\begin{aligned}\frac{\partial f}{\partial t} &= 2f\left(\frac{h'}{h}+\frac{\phi''}{\phi'}+\frac{\phi_r'}{\phi_r}\right)+\frac{1}{h'}\left[-2ff_{rr}-\frac{f}{h}h'^2-f_r^2-\frac{f}{h}f_rh_r-2\frac{f}{r}f_r+\frac{f^2}{rh}h_r\right.\\ &\quad\left.+\frac{8\pi}{M_p^2}\left(2hf^3\phi'^2-2h^2f^2\phi_r^2-4hf^2V(\phi)\right)\right] = f'\end{aligned} \tag{25}$$

$$\frac{\partial\phi'}{\partial t} = \frac{f}{h}\phi_{rr}+\frac{2hf}{r}\phi_r-fV'(\phi)+\frac{3f}{2}h_r\phi_r+\frac{h}{2}f_r\phi_r-\frac{f}{2h}\phi'h'+\frac{3f}{2}\phi'f' \tag{26}$$

Notably, the r derivatives of h and f are

$$\frac{\partial h}{\partial r} = \frac{h}{r}(1-h)+\frac{8\pi}{M_p^2}\left[\frac{rh^2}{f}\phi'^2-\frac{1}{2}rfh^2\phi'^2+\frac{1}{2}rh^3\phi_r^2+rh^2V(\phi)\right] \tag{27}$$

$$\frac{\partial f}{\partial r} = -\frac{f}{r}(1-h)+\frac{8\pi}{M_p^2}\left[rf\phi_r^2+\frac{1}{2}rf^2h\phi'^2-\frac{1}{2}rfh^2\phi_r^2-rfhV(\phi)\right] \tag{28}$$

Which gives us

$$\frac{\partial f}{\partial r} = f\left(\frac{8\pi}{M_p^2}r\left(\phi_r^2+\frac{h}{f}\phi'^2\right)-\frac{h_r}{h}\right) \tag{29}$$

So, it should follow that

$$f = h^{-1} e^{\frac{8\pi}{M_p^2}\int_0^r r\left(\phi_r^2 + \frac{h}{f}\phi'^2\right)dr} \tag{30}$$

Provided that we are considering small r values, and that ϕ' and ϕ_r are small relative to the plank mass, it appears that f differes little from h^{-1}.

On the other hand, if we substitute f' in the ϕ'' equation, we can get

$$\begin{aligned}(1-3f^2)\frac{\partial\phi'}{\partial t} &= \frac{f}{h}\phi_{rr} + \frac{2hf}{r}\phi_r - fV'(\phi) + \frac{3f}{2}h_r\phi_r + \frac{h}{2}f_r\phi_r - \frac{f}{2h}\phi' h' \\ &+3f^2\phi'\left(\frac{h'}{h} + \frac{\phi_r'}{\phi_r}\right) + \frac{3f\phi'}{2h'}\left[-2ff_{rr} - \frac{f}{h}h'^2 - f_r^2 - \frac{f}{h}f_r h_r - 2\frac{f}{r}f_r + \frac{f^2}{rh}h_r \right. \\ &\left. +\frac{8\pi}{M_p^2}\left(2hf^3\phi'^2 - 2h^2f^2\phi_r^2 - 4hf^2V(\phi)\right)\right]\end{aligned} \tag{31}$$

Likewise, we can substitute the ϕ'' equation into the f' equation to get

$$\begin{aligned}(1-3f^2)\frac{\partial f}{\partial t} &= 2f\left(\frac{h'}{h} + \frac{\phi_r'}{\phi_r}\right) + \frac{1}{h'}\left[-2ff_{rr} - \frac{f}{h}h'^2 - f_r^2 - \frac{f}{h}f_r h_r - 2\frac{f}{r}f_r + \frac{f^2}{rh}h_r \right. \\ &\left. +\frac{8\pi}{M_p^2}\left(2hf^3\phi'^2 - 2h^2f^2\phi_r^2 - 4hf^2V(\phi)\right)\right] \\ &+\frac{2f}{\phi'}\left(\frac{f}{h}\phi_{rr} + \frac{2hf}{r}\phi_r - fV'(\phi) + \frac{3f}{2}h_r\phi_r + \frac{h}{2}f_r\phi_r - \frac{f}{2h}\phi' h'\right)\end{aligned} \tag{32}$$