This is a calculation of the Einstein's field equations for a general radially symmetric scalar field. The line element is chosen to be

$$ds^2 = -dt^2 + A(r,t)^2 dr^2 + B(r,t)^2 r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

The ϕ(scalar field) variation equation is

$$-\ddot{\phi} + A^{-2}\phi'' - \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\dot{\phi} + A^{-2}\left(-\frac{A'}{A} + 2\frac{B'}{B} + \frac{2}{r}\right)\phi' - \frac{\partial U}{\partial \phi} = 0 \tag{2}$$

Here, we build up the field equations. From here on, "′" denotes a partial derivative with respect to r, and the over-dot denotes a partial derivative with respect to t. The the connection, represented as a series of matrices, is

$$\Gamma^0_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \dot{A}A & 0 & 0 \\ 0 & 0 & \dot{B}Br^2 & 0 \\ 0 & 0 & 0 & \dot{B}Br^2\sin^2\theta \end{pmatrix} \tag{3}$$

$$\Gamma^1_{\mu\nu} = \begin{pmatrix} 0 & \frac{\dot{A}}{A} & 0 & 0 \\ \frac{\dot{A}}{A} & \frac{A'}{A} & 0 & 0 \\ 0 & 0 & -\frac{B'B}{A^2}r^2 - \frac{B^2}{A^2}r & 0 \\ 0 & 0 & 0 & \left(-\frac{B'B}{A^2}r^2 - \frac{B^2}{A^2}r\right)\sin^2\theta \end{pmatrix} \tag{4}$$

$$\Gamma^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{\dot{B}}{B} & 0 \\ 0 & 0 & \frac{B'}{B} + \frac{1}{r} & 0 \\ \frac{\dot{B}}{B} & \frac{B'}{B} + \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{pmatrix} \tag{5}$$

$$\Gamma^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{\dot{B}}{B} \\ 0 & 0 & 0 & \frac{B'}{B} + \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ \frac{\dot{B}}{B} & \frac{B'}{B} + \frac{1}{r} & \cot\theta & 0 \end{pmatrix} \tag{6}$$

We use these to find the curvature tensor, $R_{\mu\nu}$.

$$\partial_\lambda \Gamma^\lambda_{\mu\nu} = \begin{pmatrix} 0 & \frac{\dot{A}'}{A} - \frac{\dot{A}A'}{A^2} & 0 & 0 \\ \frac{\dot{A}'}{A} - \frac{\dot{A}A'}{A^2} & \frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2} + \frac{A''}{A} - \frac{A'^2}{A^2} & 0 & 0 \\ 0 & 0 & (\ddot{B}B + \dot{B}^2)r^2 - \left(\frac{B'B}{A^2}r^2 + \frac{B^2}{A^2}r\right)' & 0 \\ 0 & 0 & 0 & -\cos^2\theta + \sin^2\theta\left(1 + \partial_\lambda\Gamma^\lambda_{22}\right) \end{pmatrix} \tag{7}$$

$$\left(\frac{B'B}{A^2}r^2 + \frac{B^2}{A^2}r\right)' = \frac{B''B}{A^2}r^2 + \frac{B'^2}{A^2}r^2 - 2\frac{B'BA'}{A^3}r^2 + 4\frac{B'B}{A^2}r - 2\frac{A'B^2}{A^3}r + \frac{B^2}{A^2} \tag{8}$$

$$\partial_\nu \Gamma^\lambda_{\mu\lambda} = \begin{pmatrix} \frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2} + 2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} & \frac{\dot{A}'}{A} - \frac{\dot{A}A'}{A^2} + 2\frac{\dot{B}B'}{B} - \frac{\dot{B}B'}{B^2} & 0 & 0 \\ \frac{\dot{A}'}{A} - \frac{\dot{A}A'}{A^2} + 2\frac{\dot{B}B'}{B} - \frac{\dot{B}B'}{B^2} & \frac{A''}{A} - \frac{A'^2}{A^2} + 2\frac{B'}{B} - \frac{B'^2}{B^2} - \frac{2}{r^2} & 0 & 0 \\ 0 & 0 & -\csc^2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

$$
\Gamma^{\lambda}_{\rho\lambda}\Gamma^{\rho}_{\mu\nu} = \begin{pmatrix} 0 & \left(\frac{A'}{A}+2\frac{B'}{B}+\frac{2}{r^2}\right)\frac{\dot{A}}{A} & \cot\theta\frac{\dot{B}}{B} & 0 \\ \left(\frac{A'}{A}+2\frac{B'}{B}+\frac{2}{r^2}\right)\frac{\dot{A}}{A} & \dot{A}^2+2\dot{A}A\frac{\dot{B}}{B}+\left(\frac{A'}{A}+2\frac{B'}{B}+\frac{2}{r}\right)\frac{A'}{A} & \left(\frac{B'}{B}+\frac{1}{r}\right)\cot\theta & 0 \\ \cot\theta\frac{\dot{B}}{B} & \left(\frac{B'}{B}+\frac{1}{r}\right)\cot\theta & \left(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right)\dot{B}Br^2-\left(\frac{A'}{A}+2\frac{B'}{B}+\frac{2}{r}\right)\left(\frac{B'}{B}+\frac{1}{r}\right)\frac{B^2}{A^2}r^2 & 0 \\ 0 & 0 & 0 & \Gamma^{\lambda}_{\rho\lambda}\Gamma^{\rho}_{22}\sin^2\theta-\cos^2\theta \end{pmatrix} \tag{10}
$$

$$
\Gamma^{\lambda}_{\nu\rho}\Gamma^{\rho}_{\mu\lambda} = \begin{pmatrix} \frac{\dot{A}^2}{A^2}+2\frac{\dot{B}^2}{B} & \frac{\dot{A}A'}{A^2}+2\frac{\dot{B}B'}{B^2}+\frac{2}{r}\frac{\dot{B}}{B} & \cot\theta\frac{\dot{B}}{B} & 0 \\ \frac{\dot{A}A'}{A^2}+2\frac{\dot{B}B'}{B^2}+\frac{2}{r}\frac{\dot{B}}{B} & 2\dot{A}^2+\frac{A'^2}{A^2}+2\left(\frac{B'}{B}+\frac{1}{r}\right)^2 & \left(\frac{B'}{B}+\frac{1}{r}\right)\cot\theta & 0 \\ \cot\theta\frac{\dot{B}}{B} & \left(\frac{B'}{B}+\frac{1}{r}\right)\cot\theta & \cot^2\theta-2\left(\frac{B'}{B}+\frac{1}{r}\right)^2\frac{B^2}{A^2}r^2+2\dot{B}^2r^2 & 0 \\ 0 & 0 & 0 & \Gamma^{\lambda}_{2\rho}\Gamma^{\rho}_{2\lambda}\sin^2\theta \end{pmatrix} \tag{11}
$$

$$
\begin{aligned}
R_{00} &= -\frac{\ddot{A}}{A}-2\frac{\ddot{B}}{B} \\
R_{11} &= \ddot{A}A+2\dot{A}A\frac{\dot{B}}{B}-2\frac{B''}{B}+\frac{2}{r}\left(\frac{A'}{A}-2\frac{B'}{B}\right)+2\frac{A'B'}{AB} \\
R_{01} &= \frac{2}{r}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)+2\frac{\dot{A}B'}{AB}-\frac{\dot{B}'}{B} \\
R_{22} &= B^2r^2\left(\frac{1}{r^2B^2}-\frac{1}{r^2A^2}+\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}^2}{B^2}+\frac{\ddot{B}}{B}-\frac{B''}{A^2B}-\frac{B'^2}{A^2B^2}+\frac{A'B'}{A^3B}+\frac{A'}{rA^3}-4\frac{B'}{rA^2B}\right) \\
R_{33} &= R_{22}\sin^2\theta
\end{aligned}
$$

$$
R = -R_{00}+A^{-2}R_{11}+2B^{-2}r^{-2}R_{22} \tag{12}
$$

$$
\begin{aligned}
R \quad = \quad & \frac{\ddot{A}}{A}+2\frac{\ddot{B}}{B}+\frac{\ddot{A}}{A}+2\frac{\dot{A}\dot{B}}{AB}-2\frac{B''}{A^2B}+\frac{2}{rA^2}\left(\frac{A'}{A}-2\frac{B'}{B}\right)+2\frac{A'B'}{A^3B}+\frac{2}{r^2}\left(\frac{1}{B^2}-\frac{1}{A^2}\right) \\
& +2\frac{\dot{A}\dot{B}}{AB}+2\frac{\dot{B}^2}{B^2}+2\frac{\ddot{B}}{B}-2\frac{B''}{A^2B}-2\frac{B'^2}{A^2B^2}+2\frac{A'B'}{A^3B}+2\frac{A'}{rA^3}-8\frac{B'}{rA^2B}
\end{aligned}
$$

$$
R = 2\frac{\ddot{A}}{A}+4\frac{\ddot{B}}{B}+2\frac{\dot{B}^2}{B^2}+4\frac{\dot{A}\dot{B}}{AB}-4\frac{B''}{A^2B}+4\frac{A'B'}{A^3B}-2\frac{B'^2}{A^2B^2}+4\frac{A'}{rA^3}-12\frac{B'}{rA^2B}+\frac{2}{r^2}\left(\frac{1}{B^2}-\frac{1}{A^2}\right) \tag{13}
$$

The Einstein's tensor is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = G_{\mu\nu} = \frac{8\pi}{M_p^2} T_{\mu\nu} \tag{14}$$

$$G_{00} = R_{00} - \frac{1}{2} g_{00} R = \frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}\dot{B}}{AB} - 2\frac{B''}{A^2 B} + 2\frac{A'B'}{A^3 B} - \frac{B'^2}{A^2 B^2} + 2\frac{A'}{rA^3} - 6\frac{B'}{rA^2 B} + \frac{1}{r^2}\left(\frac{1}{B^2} - \frac{1}{A^2}\right) \tag{15}$$

$$G_{11} = R_{11} - \frac{1}{2} g_{11} R = -A^2 \left[-2\frac{B'}{rBA^2} + 2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{B'^2}{A^2 B^2} + \frac{1}{r^2}\left(\frac{1}{B^2} - \frac{1}{A^2}\right)\right] \tag{16}$$

$$G_{22} = R_{22} - \frac{1}{2} g_{22} R = -B^2 r^2 \left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{B''}{A^2 B} + \frac{A'B'}{A^3 B} + \frac{A'}{rA^3} - 2\frac{B'}{rA^2 B}\right) \tag{17}$$

The stress-energy tensor for this metric is

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi)\right) \tag{18}$$

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(-\frac{1}{2}\dot{\phi}^2 + \frac{1}{2} A^{-2} \phi'^2 + V(\phi)\right) \tag{19}$$

So, putting everything together and simplifying the 11 and 22 equations, we have

$$-2\frac{B'}{rBA^2} + 2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{B'^2}{A^2 B^2} + \frac{1}{r^2}\left(\frac{1}{B^2} - \frac{1}{A^2}\right) = -\frac{8\pi}{M_;^2}\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2} A^{-2} \phi'^2 - V(\phi)\right) \tag{20}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{B''}{A^2 B} + \frac{A'B'}{A^3 B} + \frac{A'}{rA^3} - 2\frac{B'}{rA^2 B} = -\frac{8\pi}{M_p^2}\left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2} A^{-2} \phi'^2 - V(\phi)\right) \tag{21}$$

So,

$$\frac{\ddot{B}}{B} = \frac{8\pi}{M_p^2}\left(-\frac{1}{4}\dot{\phi}^2 - \frac{1}{4} A^{-2} \phi'^2 + \frac{1}{2} V(\phi)\right) + \frac{B'}{rBA^2} - \frac{\dot{B}^2}{2B^2} + \frac{B'^2}{2A^2 B^2} - \frac{1}{2r^2}\left(\frac{1}{B^2} - \frac{1}{A^2}\right) \tag{22}$$

$$\frac{\ddot{A}}{A} = \frac{8\pi}{M_p^2}\left(-\frac{1}{4}\dot{\phi}^2 + \frac{3}{4} A^{-2} \phi'^2 + \frac{1}{2} V(\phi)\right) - \frac{\dot{A}\dot{B}}{AB} + \frac{B''}{A^2 B} - \frac{A'B'}{A^3 B} - \frac{A'}{rA^3} + \frac{B'}{rA^2 B} + \frac{\dot{B}^2}{2B^2} - \frac{B'^2}{2A^2 B^2} + \frac{1}{2r^2}\left(\frac{1}{B^2} - \frac{1}{A^2}\right) \tag{23}$$

We also have an easily accessible equation from the 01 component:

$$\frac{2}{r}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) + 2\frac{\dot{A}B'}{AB} - \frac{\dot{B}'}{B} = \frac{8\pi}{M_p^2}\dot{\phi}\phi' \tag{24}$$

and the 00 component comes out as

$$\frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}\dot{B}}{AB} - 2\frac{B''}{A^2 B} + 2\frac{A'B'}{A^3 B} - \frac{B'^2}{A^2 B^2} + 2\frac{A'}{rA^3} - 6\frac{B'}{rA^2 B} + \frac{1}{r^2}\left(\frac{1}{B^2} - \frac{1}{A^2}\right) = \frac{8\pi}{M_p^2}\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2} A^{-2} \phi'^2 + V(\phi)\right) \tag{25}$$

This is in agreement with Cho and Vilenkin, gr-qc/9708005