Field equations for a general radially symmetric scalar field. The line element is

$$ds^2 = -C^2(r,t)dt^2 + A^2(r,t)dr^2 + B^2(r,t)r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

The ϕ(scalar field) variation equation is

$$-C^{-2}\ddot{\phi} + A^{-2}\phi'' - C^{-2}\left(-\frac{\dot{C}}{C} + \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\dot{\phi} + A^{-2}\left(\frac{C'}{C} - \frac{A'}{A} + 2\frac{B'}{B} + \frac{2}{r}\right)\phi' - \frac{\partial U}{\partial \phi} = 0 \tag{2}$$

Building up the field equations. From here on, ’′ denotes a partial derivative with respect to t, and $_r$ denotes a partial derivative with respect to r.

$$\Gamma^0_{\mu\nu} = \begin{pmatrix} \frac{\dot{C}}{C} & \frac{C'}{C} & 0 & 0 \\ \frac{C'}{C} & \frac{\dot{A}A}{C^2} & 0 & 0 \\ 0 & 0 & \frac{\dot{B}B}{C^2}r^2 & 0 \\ 0 & 0 & 0 & \frac{\dot{B}B}{C^2}r^2\sin^2\theta \end{pmatrix} \tag{3}$$

$$\Gamma^1_{\mu\nu} = \begin{pmatrix} \frac{C'C}{A^2} & \frac{\dot{A}}{A} & 0 & 0 \\ \frac{\dot{A}}{A} & \frac{A'}{A} & 0 & 0 \\ 0 & 0 & -\frac{(B^2r^2)'}{2A^2} & 0 \\ 0 & 0 & 0 & -\frac{(B^2r^2)'}{2A^2}\sin^2\theta \end{pmatrix} \tag{4}$$

$$\Gamma^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{\dot{B}}{B} & 0 \\ 0 & 0 & \frac{(Br)'}{Br} & 0 \\ \frac{\dot{B}}{B} & \frac{(Br)'}{Br} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{pmatrix} \tag{5}$$

$$\Gamma^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{\dot{B}}{B} \\ 0 & 0 & 0 & \frac{(Br)'}{Br} \\ 0 & 0 & 0 & \cot\theta \\ \frac{\dot{B}}{B} & \frac{(Br)'}{Br} & \cot\theta & 0 \end{pmatrix} \tag{6}$$

$$\partial_\lambda \Gamma^\lambda_{\mu\nu} = \begin{pmatrix} \left(\dot{\frac{\dot{C}}{C}}\right) + \left(\frac{C'C}{A^2}\right)' & \left(\frac{\dot{A}}{A}\right)' + \left(\frac{\dot{C}}{C}\right)' & 0 & 0 \\ \left(\frac{\dot{A}}{A}\right)' + \left(\frac{\dot{C}}{C}\right)' & \left(\dot{\frac{\dot{A}A}{C^2}}\right) + \left(\frac{A'}{A}\right)' & 0 & 0 \\ 0 & 0 & \left(\dot{\frac{\dot{B}B}{C^2}}\right) r^2 - \left(\frac{(B^2r^2)'}{2A^2}\right)' & 0 \\ 0 & 0 & 0 & -\cos^2\theta + \sin^2\theta + \left(\partial_\lambda \Gamma^\lambda_{22}\right)\sin^2\theta \end{pmatrix} \tag{7}$$

$$\left(\frac{(B^2r^2)'}{2A^2}\right)' = \frac{B''B}{A^2}r^2 + \frac{B'^2}{A^2}r^2 - 2\frac{B'BA'}{A^3}r^2 + 4\frac{B'B}{A^2}r - 2\frac{A'B^2}{A^3}r + \frac{B^2}{A^2} \tag{8}$$

$$\partial_\nu \Gamma^\lambda_{\mu\lambda} = \begin{pmatrix} \left(\dot{\frac{\dot{C}}{C}}\right) + \left(\dot{\frac{\dot{A}}{A}}\right) + 2\left(\dot{\frac{\dot{B}}{B}}\right) & \left(\frac{\dot{C}}{C}\right)' + \left(\frac{\dot{A}}{A}\right)' + 2\left(\frac{\dot{B}}{B}\right)' & 0 & 0 \\ \left(\frac{\dot{C}}{C}\right)' + \left(\frac{\dot{A}}{A}\right)' + 2\left(\frac{\dot{B}}{B}\right)' & \left(\frac{C'}{C}\right)' + \left(\frac{A'}{A}\right)' + 2\left(\frac{(Br)'}{Br}\right)' & 0 & 0 \\ 0 & 0 & -1 - \cot^2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

$$\Gamma^\lambda_{\rho\lambda}\Gamma^\rho_{\mu\nu} = \begin{pmatrix} \left(\frac{\dot{C}}{C} + \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\frac{\dot{C}}{C} + \left(\frac{C'}{C} + \frac{A'}{A} + 2\frac{(Br)'}{Br}\right)\frac{C'C}{A^2} & \left(\frac{\dot{C}}{C} + \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\frac{C'}{C} + \left(\frac{C'}{C} + \frac{A'}{A} + 2\frac{(Br)'}{Br}\right)\frac{\dot{A}}{A} & \frac{\dot{B}}{B}\cot\theta & 0 \\ \left(\frac{\dot{C}}{C} + \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\frac{C'}{C} + \left(\frac{C'}{C} + \frac{A'}{A} + 2\frac{(Br)'}{Br}\right)\frac{\dot{A}}{A} & \left(\frac{\dot{C}}{C} + \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\frac{\dot{A}A}{C^2} + \left(\frac{C'}{C} + \frac{A'}{A} + 2\frac{(Br)'}{Br}\right)\frac{A'}{A} & \frac{(Br)'}{Br}\cot\theta & 0 \\ \frac{\dot{B}}{B}\cot\theta & \frac{(Br)'}{Br}\cot\theta & \left(\frac{\dot{C}}{C} + \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\frac{\dot{B}B}{C^2}r^2 - \left(\frac{C'}{C} + \frac{A'}{A} + 2\frac{(Br)'}{Br}\right)\frac{(B^2r^2)'}{2A^2} & 0 \\ 0 & 0 & 0 & \Gamma^\lambda_{\rho\lambda}\Gamma^\rho_{22}\sin^2\theta - \cos^2\theta \end{pmatrix} \tag{10}$$

$$\Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\lambda} = \begin{pmatrix} \frac{\dot{C}^2}{C^2} + 2\frac{C'^2}{A^2} + \frac{\dot{A}^2}{A^2} + 2\frac{\dot{B}^2}{B^2} & \frac{\dot{C}C'}{C^2} + 2\frac{\dot{A}C'}{AC} + \frac{\dot{A}A'}{A^2} + 2\frac{\dot{B}(Br)'}{B^2r} & \frac{\dot{B}}{B}\cot\theta & 0 \\ \frac{\dot{C}C'}{C^2} + 2\frac{\dot{A}C'}{AC} + \frac{\dot{A}A'}{A^2} + 2\frac{\dot{B}(Br)'}{B^2r} & \frac{C'^2}{C^2} + 2\frac{\dot{A}^2}{C^2} + \frac{A'^2}{A^2} + 2\left(\frac{(Br)'}{Br}\right)^2 & \frac{(Br)'}{Br}\cot\theta & 0 \\ \frac{\dot{B}}{B}\cot\theta & \frac{(Br)'}{Br}\cot\theta & \cot^2\theta - 2\left(\frac{(Br)'}{A}\right)^2 + 2\frac{\dot{B}^2r^2}{C^2} & 0 \\ 0 & 0 & 0 & \Gamma^\lambda_{2\rho}\Gamma^\rho_{2\lambda}\sin^2\theta - 3\cos^2\theta \end{pmatrix} \tag{11}$$

$$R_{00} = \left(\frac{\dot{\dot{C}}}{C}\right) + \left(\frac{C'C}{A^2}\right)' - \left(\frac{\dot{\dot{C}}}{C}\right) - \left(\frac{\dot{\dot{A}}}{A}\right) - 2\left(\frac{\dot{\dot{B}}}{B}\right) + \left(\frac{\dot{C}}{C} + \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\frac{\dot{C}}{C} + \left(\frac{C'}{C} + \frac{A'}{A} + 2\frac{(Br)'}{Br}\right)\frac{C'C}{A^2} - \frac{\dot{C}^2}{C^2} - 2\frac{C'^2}{A^2} - \frac{\dot{A}^2}{A^2} - 2\frac{\dot{B}^2}{B^2}$$

$$R_{11} = \left(\frac{\dot{\dot{A}}A}{C^2}\right) + \left(\frac{A'}{A}\right)' - \left(\frac{C'}{C}\right)' - \left(\frac{A'}{A}\right)' - 2\left(\frac{(Br)'}{Br}\right)' + \left(\frac{\dot{C}}{C} + \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\frac{\dot{A}A}{C^2} + \left(\frac{C'}{C} + \frac{A'}{A} + 2\frac{(Br)'}{Br}\right)\frac{A'}{A} - \frac{C'^2}{C^2} - 2\frac{\dot{A}^2}{C^2} - \frac{A'^2}{A^2} - 2\left(\frac{(Br)'}{Br}\right)^2$$

$$R_{01} = -\left(\frac{\dot{A}}{A}\right)' + \left(\frac{\dot{C}}{C}\right)' - \left(\frac{\dot{C}}{C}\right)' - \left(\frac{\dot{A}}{A}\right)' - 2\left(\frac{\dot{B}}{B}\right)' + \left(\frac{\dot{C}}{C} + \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\frac{C'}{C} + \left(\frac{C'}{C} + \frac{A'}{A} + 2\frac{(Br)'}{Br}\right)\frac{\dot{A}}{A} - \frac{\dot{C}C'}{C^2} - 2\frac{\dot{A}C'}{AC} - \frac{\dot{A}A'}{A^2} - 2\frac{\dot{B}(Br)'}{B^2r}$$

$$R_{22} = \left(\frac{\dot{\dot{B}}B}{C^2}\right)r^2 - \left(\frac{(B^2r^2)'}{2A^2}\right)' + 1 + \cot^2\theta + \left(\frac{\dot{C}}{C} + \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\frac{\dot{B}B}{C^2}r^2 - \left(\frac{C'}{C} + \frac{A'}{A} + 2\frac{(Br)'}{Br}\right)\frac{(B^2r^2)'}{2A^2} - \cot^2\theta + 2\left(\frac{(Br)'}{A}\right)^2 - 2\frac{\dot{B}^2r^2}{C^2}$$

$$R_{33} = -\cos^2\theta + \sin^2\theta + \left(\partial_\lambda \Gamma^\lambda_{22}\right)\sin^2\theta + \Gamma^\lambda_{\rho\lambda}\Gamma^\rho_{22}\sin^2\theta - \cos^2\theta - \Gamma^\lambda_{2\rho}\Gamma^\rho_{2\lambda}\sin^2\theta + 3\cos^2\theta$$

$$R_{00} = -C^2\left[-\frac{C''}{A^2C} + \frac{\ddot{A}}{AC^2} + 2\frac{\ddot{B}}{BC^2} - \frac{\dot{A}\dot{C}}{AC^3} - 2\frac{\dot{B}\dot{C}}{BC^3} + \frac{A'C'}{A^3C} - 2\frac{(Br)'}{Br}\frac{C'}{A^2C}\right]$$

$$R_{11} = A^2\left[\frac{\ddot{A}}{AC^2} - \frac{C''}{A^2C} - 2\frac{(Br)''}{A^2Br} - \frac{\dot{A}\dot{C}}{AC^3} + 2\frac{\dot{A}\dot{B}}{ABC^2} + \frac{A'C'}{A^3C} + 2\frac{(Br)'}{Br}\frac{A'}{A^3}\right]$$

$$R_{01} = -2\frac{\dot{B}'}{B} + 2\frac{\dot{B}}{B}\frac{C'}{C} + 2\frac{(Br)'}{Br}\frac{\dot{A}}{A} - 2\frac{\dot{B}}{Br}$$

$$R_{22} = B^2r^2\left[\frac{1}{B^2r^2} + \frac{\ddot{B}}{BC^2} - \frac{(Br)''}{A^2Br} + \left(-\frac{\dot{C}}{C} + \frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\frac{\dot{B}}{BC^2} + \left(-\frac{C'}{C} + \frac{A'}{A} - \frac{(Br)'}{Br}\right)\frac{(Br)'}{A^2Br}\right]$$

$$R_{33} = B^2r^2\sin^2\theta\left[\frac{1}{B^2r^2} + \frac{\ddot{B}}{BC^2} - \frac{(Br)''}{A^2Br} + \left(-\frac{\dot{C}}{C} + \frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\frac{\dot{B}}{BC^2} + \left(-\frac{C'}{C} + \frac{A'}{A} - \frac{(Br)'}{Br}\right)\frac{(Br)'}{A^2Br}\right]$$

$$\begin{aligned} R \quad = \quad & -\frac{C''}{A^2C} + \frac{\ddot{A}}{AC^2} + 2\frac{\ddot{B}}{BC^2} - \frac{\dot{A}\dot{C}}{AC^3} - 2\frac{\dot{B}\dot{C}}{BC^3} + \frac{A'C'}{A^3C} - 2\frac{(Br)'}{Br}\frac{C'}{A^2C} \\ & + \frac{\ddot{A}}{AC^2} - \frac{C''}{A^2C} - 2\frac{(Br)''}{A^2Br} - \frac{\dot{A}\dot{C}}{AC^3} + 2\frac{\dot{A}\dot{B}}{ABC^2} + \frac{A'C'}{A^3C} + 2\frac{(Br)'}{Br}\frac{A'}{A^3} \\ & + 2\left[\frac{1}{B^2r^2} + \frac{\ddot{B}}{BC^2} - \frac{(Br)''}{A^2Br} + \left(-\frac{\dot{C}}{C} + \frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\frac{\dot{B}}{BC^2} + \left(-\frac{C'}{C} + \frac{A'}{A} - \frac{(Br)'}{Br}\right)\frac{(Br)'}{A^2Br}\right] \end{aligned}$$

$$R = -2\frac{C''}{A^2C} + 2\frac{\ddot{A}}{AC^2} + 4\frac{\ddot{B}}{BC^2} - 2\frac{\dot{A}\dot{C}}{AC^3} - 4\frac{\dot{B}\dot{C}}{BC^3} + 2\frac{A'C'}{A^3C} - 4\frac{(Br)'}{Br}\frac{C'}{A^2C} - 4\frac{(Br)''}{A^2Br} + 4\frac{\dot{A}\dot{B}}{ABC^2} + 4\frac{(Br)'}{Br}\frac{A'}{A^3} + 2\frac{\dot{B}^2}{B^2C^2} - 2\frac{(Br)'^2}{A^2B^2r^2} + \frac{2}{B^2r^2} \tag{12}$$

$$\frac{1}{2}R = -\frac{C''}{A^2C} + \frac{\ddot{A}}{AC^2} + 2\frac{\ddot{B}}{BC^2} - \frac{\dot{A}\dot{C}}{AC^3} - 2\frac{\dot{B}\dot{C}}{BC^3} + \frac{A'C'}{A^3C} - 2\frac{(Br)'}{Br}\frac{C'}{A^2C} - 2\frac{(Br)''}{A^2Br} + 2\frac{\dot{A}\dot{B}}{ABC^2} + 2\frac{(Br)'}{Br}\frac{A'}{A^3} + \frac{\dot{B}^2}{B^2C^2} - \frac{(Br)'^2}{A^2B^2r^2} + \frac{1}{B^2r^2} \tag{13}$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right) \tag{14}$$

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(-\frac{1}{2} C^{-2} \dot{\phi}^2 + \frac{1}{2} A^{-2} \phi'^2 + V(\phi) \right) \tag{15}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi}{M_p^2} T_{\mu\nu} \tag{16}$$

$$G_{00} = R_{00} - \frac{1}{2} g_{00} R = -C^2 \left[-2 \frac{(Br)''}{A^2 Br} + 2 \frac{\dot{A}\dot{B}}{ABC^2} + 2 \frac{(Br)'}{Br} \frac{A'}{A^3} + \frac{\dot{B}^2}{B^2 C^2} - \frac{(Br)'^2}{A^2 B^2 r^2} + \frac{1}{B^2 r^2} \right] \tag{17}$$

$$G_{11} = R_{11} - \frac{1}{2} g_{11} R = A^2 \left[-2 \frac{\ddot{B}}{BC^2} + 2 \frac{\dot{B}\dot{C}}{BC^3} + 2 \frac{(Br)'}{Br} \frac{C'}{A^2 C} - \frac{\dot{B}^2}{B^2 C^2} + \frac{(Br)'^2}{A^2 B^2 r^2} - \frac{1}{B^2 r^2} \right] \tag{18}$$

$$G_{22} = R_{22} - \frac{1}{2} g_{22} R = B^2 r^2 \left[\frac{C''}{A^2 C} - \frac{\ddot{A}}{AC^2} - \frac{\ddot{B}}{BC^2} + \frac{\dot{A}\dot{C}}{AC^3} + \frac{\dot{B}\dot{C}}{BC^3} - \frac{A'C'}{A^3 C} + \frac{(Br)'}{Br} \frac{C'}{A^2 C} + \frac{(Br)''}{A^2 Br} - \frac{\dot{A}\dot{B}}{ABC^2} - \frac{(Br)'}{Br} \frac{A'}{A^3} \right] \tag{19}$$

$$G_{33} = \sin^2 \theta G_{22} \tag{20}$$

Then the 00 equation is

$$-2\frac{(Br)''}{A^2Br}+2\frac{\dot{A}\dot{B}}{ABC^2}+2\frac{(Br)'}{Br}\frac{A'}{A^3}+\frac{\dot{B}^2}{B^2C^2}-\frac{(Br)'^2}{A^2B^2r^2}+\frac{1}{B^2r^2}=\frac{8\pi}{M_p^2}\left[\frac{\dot{\phi}^2}{2C^2}+\frac{\phi'^2}{2A^2}+V(\phi)\right] \tag{21}$$

The 11 equation is

$$-2\frac{\ddot{B}}{BC^2}+2\frac{\dot{B}\dot{C}}{BC^3}+2\frac{(Br)'}{Br}\frac{C'}{A^2C}-\frac{\dot{B}^2}{B^2C^2}+\frac{(Br)'^2}{A^2B^2r^2}-\frac{1}{B^2r^2}=\frac{8\pi}{M_p^2}\left[\frac{\dot{\phi}^2}{2C^2}+\frac{\phi'^2}{2A^2}-V(\phi)\right] \tag{22}$$

The 22 and 33 equations are both

$$\frac{C''}{A^2C}-\frac{\ddot{A}}{AC^2}-\frac{\ddot{B}}{BC^2}+\frac{\dot{A}\dot{C}}{AC^3}+\frac{\dot{B}\dot{C}}{BC^3}-\frac{A'C'}{A^3C}+\frac{(Br)'}{Br}\frac{C'}{A^2C}+\frac{(Br)''}{A^2Br}-\frac{\dot{A}\dot{B}}{ABC^2}-\frac{(Br)'}{Br}\frac{A'}{A^3}=\frac{8\pi}{M_p^2}\left[\frac{\dot{\phi}^2}{2C^2}-\frac{\phi'^2}{2A^2}-V(\phi)\right] \tag{23}$$

and the 01 equation is

$$-2\frac{\dot{B}'}{B}+2\frac{\dot{B}}{B}\frac{C'}{C}+2\frac{(Br)'}{Br}\frac{\dot{A}}{A}-2\frac{\dot{B}}{Br}=\frac{8\pi}{M_p^2}\dot{\phi}\phi' \tag{24}$$

Restating, the scalar field equation is

$$-\frac{\ddot{\phi}}{C^2}+\frac{\phi''}{A^2}-\left(-\frac{\dot{C}}{C}+\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right)\frac{\dot{\phi}}{C^2}+\left(\frac{C'}{C}-\frac{A'}{A}+2\frac{B'}{B}+\frac{2}{r}\right)\frac{\phi'}{A^2}-\frac{\partial U}{\partial\phi}=0 \tag{25}$$